be taken at pleasure and the angles that they form with each other are entirely arbitrary. A superficial view of this theorem might lead one to think that the figures of axonometry may be drawn haphazard and that they will then represent the space figures in mind from some point of vision. On closer acquaintance, however, the theorem is seen to be associated with law, rather than license. It relates to setting up in the plane figure a standard of measurement which makes for definiteness.

RECENT PUBLICATIONS.

REVIEWS.

Girolamo Saccheri's Euclides Vindicatus. Edited and translated by GEORGE BRUCE HALSTED. Chicago and London, Open Court, 1920. 8vo. 30 + 246 pp. Price $2.00.

The Jesuit Girolamo Saccheri was born in 1667 and first taught in a college of his order in Milan where Tommaso Ceva, a brother of the Giovanni Ceva whose triangle theorem is well known, was teacher of mathematics. The influence of these brothers is apparent in the first two mathematical works which Saccheri published: (1) *Quœsita geometrica a Comite Rugerio de Vigintimilliis omnibus proposita, ab Hieronymo Saccherio Genuensi Societatis Jesu soluta.* Mediolani, 1693 (37 pages); another edition Parma, 1694 (dealing mainly with the solution of six problems in conic sections). (2) *Neostatica auctore Hieronymo Saccherio e Societate Jesu Excellentissimo senatui Mediolanensi dicata.* Mediolani, 1708 (168 pages) (discussing questions of statics and dynamics).

His third mathematical work *Euclides ab omni nævo vindicatus* was published at Milan in 1733 (16 + 142 pages + 6 plates). Saccheri died in October, 1733, and there is doubt as to whether he lived to see his completed master work issue from the press. While the work is frequently referred to during the next one hundred and fifty years it was not till the publication of an article by Beltrami[1] in 1889 that Saccheri became generally recognized as a forerunner of Legendre, Lobatchevsky, and Bolyai. Indeed Saccheri gave many of their propositions. For example, in his discussions of the parallel postulate, 1794–1833, Legendre proved, by using only the first twenty-eight propositions of Euclid's Elements, that: The sum of the angles of a triangle cannot be greater than two right angles; and that the sum must be equal to two right angles if this is true for a single triangle. Both of these propositions are proved in more general form by Saccheri.

In Saccheri's time the conception of parallels as equidistant straight lines was a favorite one, but Saccheri, like some of his predecessors, as Sommerville remarks, "saw that it would not do to assume this in the definition. He starts with two equal perpendiculars AC and BD to a line AB. When the ends C, D

[1] "Un precursore italiano di Legendre et di Lobatschewsky," *Atti della Reale Accademia dei Lincei*, Anno 1889, series 4, Vol. 5, pp. 441–448.

are joined, it is easily proved that the angles at C and D are equal; but are they right angles? Saccheri keeps an open mind, and proposes three hypotheses: (1) The hypothesis of the right angle; (2) The hypothesis of the obtuse angle; and (3) The hypothesis of the acute angle. The object of his work is to demolish the last two hypotheses and leave the first, the Euclidean hypothesis, supreme. . . . If Saccheri had had a little more imagination and been less bound down by tradition and by a firmly planted belief that Euclid's hypothesis was the only true one, he would have anticipated by a century the discovery of the two non-euclidean geometries which follow from his hypotheses of the obtuse and the acute angles."

Saccheri's work is divided into two books: the first, pages 1–101, containing the discussion indicated above; the second, a defense of the treatment of proportion found in book V of Euclid's Elements. The first complete translation of the first book was into German by Engel and Stäckel in their *Die Theorie der Parallellinien von Euklid bis auf Gauss*, Leipzig, 1895. This contains very full references to the literature of the work. An Italian translation, by G. Boccardini of both books (the first very slightly, the latter much abbreviated), appeared in the Manuali Hoepli series in 1904.

As to Dr. Halsted's translation there is no indication in the work before us that any part of the work has appeared in print elsewhere before. Nevertheless this is the case. The last paragraph of a circular advertising the book under review, and signed by Dr. Halsted, is as follows: "The English translation is a revision of the first ever made into any language, published in 1894, but long unprocurable." Hardly a single statement in this sentence is accurate. The translation in question (of propositions I–XXXVI in the first book) appeared in this MONTHLY, volumes 1–5, June, 1894–December, 1898. In "1894" the English translation of thirteen propositions only had appeared. The complete German translation of the first book was published before half of the English translation had appeared. The early numbers of the MONTHLY are not "unprocurable."

The differences between the part of Saccheri's work which appeared in this MONTHLY, and its reprint, are numerous—nearly three hundred were noticed in the first twenty-five propositions—but these have rarely introduced radical changes in the sense of the original passages. The figures are all new and are black on white instead of white on a black background. They are not repeated nearly as often as in the translation of Engel and Stäckel and the reading is consequently much less easy. In Saccheri's work the figures were on folding plates.

The additions to the original English translation are considerable: propositions 37 to 39 (the last of book 1), Saccheri's preface to the reader and synopsis of contents, and the original Latin of the whole, each page of text having the translation on the opposite page.

Dr. Halsted's "Introduction" emphasizes the importance of Saccheri's *Logica demonstrativa* first published in 1697 (the second edition appearing in 1701

and the third in 1735) in studying the *Euclides Vindicatus*. He has also appended two pages of notes, a subject index, and an index of proper names. The whole has been issued by the Open Court Publishing Company in a volume of attractive appearance.

This enterprising company and Dr. Halsted have placed mathematicians much in their debt by making the masterpieces of Saccheri,[1] Bolyai, and Lobatchevsky so readily accessible to American and English readers.

R. C. ARCHIBALD.

The Elementary Differential Geometry of Plane Curves. By R. H. FOWLER. (Cambridge Tracts in Mathematics and Mathematical Physics, No. 20.) Cambridge, at the University Press, 1920. 7 + 105 pages. Price 6 shillings.

Preface: "This tract is intended to present a precise account of the elementary differential properties of plane curves. The matter contained is in no sense new, but a suitable connected treatment in the English language has not been available.

"As a result, a number of interesting misconceptions are current in English text books. It is sufficient to mention two somewhat striking examples. (*a*) According to the ordinary definition of an envelope, as the locus of the limits of points of intersection of neighbouring curves, a curve is not the envelope of its circles of curvature, for neighbouring circles of curvature do not intersect. (*b*) The definitions of an asymptote—(1) a straight line, the distance from which of a point on the curve tends to zero as the point tends to infinity; (2) the limit of a tangent to the curve, whose point of contact tends to infinity—are not equivalent. The curve may have an asymptote according to the former definition and the tangent may exist at every point, but have no limit as its point of contact tends to infinity.

"The subjects dealt with, and the general method of treatment, are similar to those of the usual chapters on geometry in any *Cours d'Analyse*, except that in general plane curves alone are considered. At the same time extensions to three dimensions are made in a somewhat arbitrary selection of places, where the extension is immediate, and forms a natural commentary on the two dimensional work, or presents special points of interest (Frenet's formulæ). To make such extensions systematically would make the tract too long. The subject matter being wholly classical, no attempt has been made to give full references to sources of information; the reader however is referred at most stages to the analogous treatment of the subject in the *Cours* or *Traité d'analyse* of de la Vallée Poussin, Goursat, Jordan or Picard, works to which the author is much indebted.

"In general the functions, which define the curves under consideration, are (as usual) assumed to have as many continuous differential coefficients as may be mentioned. In places, however, more particularly at the beginning, this rule is deliberately departed from, and the greatest generality is sought for in the enunciation of any theorem. The determination of the *necessary and sufficient* conditions for the truth of any theorem is then the primary consideration. In the proofs of the elementary theorems, where this procedure is adopted, it is believed that this treatment will be found little more laborious than any rigorous treatment, and that it provides a connecting link between Analysis and more complicated geometrical theorems, in which insistence on the precise necessary conditions becomes tedious and out of place, and suitable sufficient conditions can always be tacitly assumed. At an earlier stage the more precise formulation of conditions may be regarded as (1) an important grounding for the student of Geometry, and (2) useful practice for the student of Analysis.

"The introductory chapter is a collection of somewhat disconnected theorems which are

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[1] For other discussions of Saccheri's *Euclides Vindicatus* the curious reader may turn to Cantor, *Vorlesungen über Geschichte der Mathematik*, Vol. 3, 2te Aufl., 1901; to P. Mansion, "Analyse des recherches du P. Saccheri, S. J., sur le postulatum d'Euclide," *Annales de la Société scientifique de Bruxelles*, 1889–1890, Vol. 14, 2d part; reprinted in a supplement to *Mathesis*, January, 1891; to G. Veronese, *Grundzüge der Geometrie*, Leipzig, 1894, pp. 636–639; and to H. S. Carslaw's English translation of Bonola's *Non-Euclidean Geometry* (Open Court, 1912).

It seems curious that there is no reference to Saccheri in *The Catholic Encyclopedia*.